Geometry and dynamics of billiards in symmetric phase space

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Abstract

The billiard problem of statistical physics is considered in a new geometric approach with a symmetric phase space. The structure and topological features of typical billiard phase portrait are defined. The connection between geometric, dynamic and statistic properties of smooth billiard is established. Other directions of the theory on development are pointed out.

1. Introduction

Billiard is one of the most important models of statistical physics and chaotic dynamics. G.D. Birkhov suggested regarding billiard as a typical conservative system [1]. A.N. Krylov based his explanation of solid spheres gas statistic properties on exponential divergence of its "billiard" trajectories [2]. In the works by Ya.G. Sinai and L.A. Bunimovich on phase trajectories mixing in scattering and defocusing billiards Boltzmann's hypothesis of molecular chaos found its further grounding [3]. Now billiard became a paradigm of deterministic chaos [4] of classical systems and is often applied [5] for the research of their quantum "twins". A lot of applied physics problems can be reduced to a billiard problem [6].

A classical billiard problem is in studying its character and distribution of its trajectories. Among the typical billiard motions one can point out the following: periodical, quasiperiodical (integrable) and irregular (chaotic) motions. Compound billiard dynamics appears in the phase portrait structure of the corresponding mapping. The latter is plotted using different geometric methods or Poincare sections. For the specification of a billiard ray it's usual to choose local Birkhov coordinates: natural parameter l in the reflection point on the border of the billiard $\partial\,\Omega$ and the incidence angle θ in the same point. They stand for canonical variables – the coordinate and the moment for Hamiltonian description of the system. Many important properties at this choice of phase space coordinates stay unnoticed. Let us choose another unifying approach. It identifies billiards with reversible mapping (with projective involution) in a symmetrical phase space. In its framework one can join together geometric, dynamic and statistic properties of billiards. Such fundamental mechanisms are analyzed in this paper.

2. Symmetric Coordinates

Let us describe geometric propagation of the rays of billiard (Fig.) as a reversible mapping B of the phase space $Z$with symmetric coordinates (z_1, z_2). The pair of these coordinates defines two successive reflections of a billiard ray from $\partial\,\Omega$. At the same time, each of the coordinates corresponds to some parameterization of the billiard border, $\vec{r}|_{\partial\Omega} = \vec{r}(z) = (x(z), y(z))$. The following topological construction appears: $Z \propto \partial\,\Omega \times \partial\,\Omega$. For a closed planar billiard one can accept $z \in S^1$ (circle) or $z \in I = [0, 1]$ the periodicity being $\vec{r}(z) = \vec{r}(z + 1)$. So we'll have a phase space as a torus $Z = T^2 = S^1 \times S^1$ or its unfolding $\Pi = I \times I$ on the plane. After each reflection of an arbitrary (incoming) billiard ray with the coordinates (z_1, z_2), we have a (reflected) ray with new coordinates (z_1', z_2'). As a result, the evolution of these successive reflections is described with a billiard cascade $(z_1, z_2) \to (z_1', z_2')$ of the form [7]

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$$B := \begin{cases} z_1' = z_2 \\ z_2' = f(z_1, z_2) \end{cases} ;\; f(f(z_1,z_2),z_2) = z_1; g(f,z_2) = g(R(z_1,z_2),z_2));\;\; R(z,z') = \frac{a(z')z + b(z')}{b(z')z - a(z')} \quad (1)$$

with the involution $f = f(z_1, z_2)$ (on the first argumentz_1), which is defined by implicit dependence on the corresponding fractional rational involution R, $R(R(z,z'),z') = z$. The coefficients $a(z) = n_x^2(z) - n_y^2(z)$; $b(z) = 2n_x^2(z)n_y^2(z)$ are expressed with (Cartesian) components of the exterior normal field $\vec{n}(z) = \vec{n}_{ext}|_{\partial\Omega}$ on the border $\partial\ \Omega$, $\vec{n} = (n_x; n_y) = (y'(z); -x'(z))$ (the stroke marks differentiation). Function g depends on the form of $\partial\ \Omega$ and $g(z_1, z_2) = g(z_2, z_1) = = [x(z_1) - x(z_2)]/[y(z_1) - y(z_2)]$. The choice of Cartesian coordinates is rather relative, because every covariant substitution $z_1 \to \xi B(z_1); z_2 \to \xi B(z_2)$ preserves the form of the mapping (1). The mapping built can be used for the description of billiard with a border of the most general type. During its derivation only the condition of elastic reflection and no restrictions of smoothness, curvature, convexity, simple connectivity of the border and so on were introduced.

The mapping (1) is invariant to the substitution, $S :=\ \ z_1 \to z_2; z_2 \to z_1$, of the incoming ray to the reflected one, i.e. $B \circ S = S \circ B$ for the composition of transformations. This means reversibility of the constructed mappings. The physical reason of this is reversibility of the system to the changes of the time sign (the direction of the motion). This is a global property. In the billiard cascade, phase trajectories with opposite directions of the motion or with opposite-directional initial rays (z_{10}, z_{20}) è $(z_{20}, z_{10})$are present simultaneously. This requirement of local reversibility is stronger. The inverse of the ray reflected with its successive reflection makes the initial incoming ray. Mathematically it leads to the appearance of a involution f in the mapping (1). The symmetry (reversibility) leads to the symmetry of the phase space and the phase portrait of the mappings (1). For every element Z there is one symmetrical to it relative to the diagonal $\Delta = \{(z_1, z_2) \in Z \mid z_1 = z_2\}$. For every function χ on Z

$$\chi(z_1, z_2) = \chi(z_2, z_1) \quad . \quad (2)$$

That's why it's natural to regard the coordinates of Z as symmetrical. This symmetricity essentially sets them apart from the variables of Hamiltionian description of the billiard. In particular, the locality of Birkhov coordinates causes explicit mappings to be obtained only for the simplest geometry $\partial\ \Omega$. In non-local coordinates, (z_1, z_2) this problem can have a general solution.

In the research of periodical trajectories of the billiard the powers of billiard mapping B^k are also used

$$B^k := \{z_1' = f_{k-1}(z,z); z_2' = f_k(z_1,z_2)\}; \quad f_k(z_1,z_2) = f(f_{k-2}(z_1,z_2), f_{k-1}(z_1,z_2)) \quad . \quad (3)$$

They include billiard "compositions" f_k, where $k = 0, 1, 2, \ldots$; $f_1 = f$; $f_0 = z_2$; $f_{-1} = z_1$. They lose the property of involution, but preserve reductibility to fractional rational transformations. The mappings (3) describe "pruned" billiard trajectories with the omission of a set of $(k-1)$ links (successive reflections).

3. Billiard Geometry: Involution Properties

All the geometric properties of a billiard are established in the specialization of mappings (1). They are concretized in the features of the involution$f$. In the appropriate (local) coordinates it can be reduced to fractional rational involution R. Projective transformations are described with fractional rational functions. Billiard is one of those transformations. In every reflection point incoming and reflected rays are joined together by a harmonic transformation G. For G projective invariant (a complex relation of four rays , incoming i, reflected r, normal n and tangent t) is equal to $(i, r, n, t) = -1$. In geometric terms involution looks the simplest

$$r = G(i; n, t); \quad G \circ G = Id \quad , \quad (4)$$

Id is an identical transformation. Let us emphasize the locality of the projective property of the billiard. The concrete form of G depends on the guiding-lines of the normal in the point of reflection, that is, on the form $\partial\,\Omega$. Harmonic mapping (4) is an involution and changes the sequence order of the ordered projective elements to the opposite. The monotony of f on the first argument is the consequence of this. This monotony of piece-wise continuous f (involution can stand discontinuity) is true for every billiard. Using the correlations (1), the involution can be laced of local branches of the form $f = g^{-1} \circ R \circ g$ (the choice of g^{-1} branch is dictated by the variational principle, the minimality of distance between the points of reflection). From that the property of monotony immediately follows

$$\partial f(z_1, z_2)/\partial z_1 < 0 \quad . \tag{5}$$

Fractional rational functions are dense everywhere in the space of continuous functions. In fact, this means the possibility of arbitrary precise approximation of different physical systems with their billiard models. This fact is used, for instance, in the analysis of energetic spectra of multi-particle systems, the description of kinetic properties of continuum (Lorenz gas model) etc. If sinus and cosine have physically appeared from the problem of oscillator, then fractional rational functions can be generated by billiard.

The reflection of ray beams from the border of the billiard can be of diffractive, focusing and neutral character. This depends on the curvature of $\partial\,\Omega$. The representation (1) gives the following property

$$sign\left\{\frac{\partial f(z_1, z_2)}{\partial z_2}\right\} = sign\left\{\hat{K}(z_2)\right\} \quad , \tag{6}$$

where $\hat{K}$ is oriented curvature in the point of reflection. For the convex border $\hat{K} > 0$ involution appears to be a monotonous function on both arguments. (For instance, for a circle, $f(z_1, z_2) = 2z_2 - z_1 \pmod 1$.) On a torus, $Z = T^2$, such involution has no breaks (Unlike everywhere dispersive billiard, $\hat{K} < 0$, with lacunas in phase space.)

Involutivity and projectivity are the main geometric properties of a billiard. The geometry (form) of its border defines the explicit form of involution. At the same time, it also defines the dynamics of the billiard.

4. Billiard Dynamics: the Structure of Symmetric Phase Space

Let us analyze the structure of symmetric phase space of a typical billiard (see Fig.). This principally solves the question of the types of dynamics and stability. For more simplicity let us regard the border $\partial\,\Omega$ as a differentiable as many times as needed. The research of billiards in polygons (without curvature) needs some peculiarities.

For high-quality research of phase portrait of the mappings and its local bifurcations normal Poincare forms are especially useful [8]. In the symmetric approach the theory of normal billiard forms appears to be the most advanced. This is connected with the flexibility (a wider class of allowable variables) of reversible systems. Any changes of variables in Hamiltonian approach are to preserve the conservation character of the mapping with the Jacobian $J = 1$ (canonical changes). Whereas the mapping (1) doesn't demand it. It Jacobian $J = -\partial f(z_1, z_2)/\partial z_1 > 0$ can take arbitrary values, $0 < J \leq 1;\ J \geq 1$. As a weak limitation, the demand for the mapping (1) to preserve measure remains. This means that $J = J(\vec{z}) = \rho(\vec{z})/\rho(B(\vec{z}))$, where $\vec{z} = (z_1, z_2);\ B\,\vec{z} = (z_2, f(z_1, z_2))$ should be true. The proof uses the equation of Frobenius–Perron for the density ρ of invariant measure (see further) and the symmetry (2) for it, $\rho(z_1, z_2) = \rho(z_2, z_1)$. This limitation can always be met preserving the main property of involution $f \circ f = id$ in new coordinates.

Omitting the details, let us present the expression for universal normal billiard form in symmetric coordinates. It is true in the neighbourhood of an arbitrary cycle of $p$order (periodic trajectory of $p$period)

$$NB^p := \begin{cases} z_1' = -\mu_{p-1} z_1 + \nu_{p-1} z_2 + z_1 P(z_1 z_2) \\ z_2' = -\mu_p z_1 + \nu_p z_2 + z_2 Q(z_1 z_2) \end{cases} \quad , \tag{7}$$

where the coefficients of the linear part are defined by the expansion of "compositions" f_{p-1} and f_p (see formula (3)) in the initial point neighbourhood of the cycle under consideration. They constitute the matrix of $\hat{L}$ linear part. Its determinant is equal to one, det $\hat{L} = 1$, that agrees with conservation of measure and the condition $J(C) = 1$ for every cycle C of mapping (1). Homogeneous polynomials P, Q (without absolute terms) define nonlinear additives. Their explicit form depends on the involution of billiard f, that is, on the form of $\partial\, \Omega$.

The character of the cycle depends on the size of trace $tr\hat{L}$. For an elliptic cycle $\left|tr\ \hat{L}\right| < 2$, for a hyperbolic one $\left|tr\ \hat{L}\right| > 2$. In the neutral case, for instance, for a billiard in a circle, $tr\hat{L} = 2$. It can be shown that for any cycle, corresponding to a periodic trajectory, passing through a concave section with concavity $\hat{K}(z_2) < 0$, $tr\hat{L} < -2$ will be true. That's why the trajectories near such cycles always are unstable and exponentially diverge from one another. Near elliptic cycles, including 2-cycles, regions of regular motion form. With the loss of ellipticity they are ruined, first forming stochastic layers and then, when the latter are covered, a chaotic sea. In reality this mechanism looks much more complicated, for example, with an intermediate pass through Cantor-tori etc. Normal forms (7) let us trace typical properties of such bifurcations, taking place when the billiard border is deformed.

The diagonal $\Delta(z_1 = z_2)$ contains all fixed points of the billiard, $B\Delta = \Delta$. This follows from the diagonal property $f(z, z) = z$ of billiard involution, resulting from its coordinate expression (1). For a convex billiard in the neighbourhood of the phase space diagonal, normal form (7) can be reduced to the mapping of a turn, i.e. a particular case of a billiard in a circle. Here the structure of elliptic and hyperbolic cycles of arbitrary high order is shown. The motion stays regular. The appearing of negative curvature ruins this situation. There is no unified transformation (or the integral of motion) near the diagonal because of appearing breaks of billiard involution.

Analytic research of the symmetric phase space structure can be continued using geometric methods. Here the important advantages of the new approach are seen. In addition to regular and chaotic components of motion the phase portrait can contain regions of forbidden motion – "lacunas" $L$and regions of degenerated motion – "discriminants" D.

Lacunas (Fig.) appear in the billiards with regions of negative curvature. They occupy the phase space part, the points of which correspond to the rays lying outside of the billiard region Ω. The coordinates of these rays meet the condition $\vec{r}(z_1) - \vec{r}(z_2) \notin \Omega$. This condition defines the inner region of lacuna L in Z. The form of the lacuna is defined by its border $\partial\, L := \{(z_1, z_2) \in Z |\, z_1 = \lambda(z_2)\}$. (Another parameterization is possible $z_2 = \lambda^{-1}(z_1)$. In this case functions $\lambda(z)$ and $\lambda^{-1}(z)$ specify the same simple closed curve, but passable in different directions. When one its branch is higher than $\Delta$and the other is lower, and vice versa.) The border $\partial\, L$ comes to the diagonal Δ transversally and crosses it twice in the points with coordinates (z_0, z_0), corresponding to the points of inflexion, $\hat{K}(z_0) = 0$.

The forbidden billiard rays (points of lacuna) lie in classically inaccessible region – geometric shade, generated by the regions $\hat{K} < 0$. The number of lacunas (on a torus $Z = T^2$) is equal to the number of negative curvature components $\partial\ \Omega$. Every lacuna is a simply connected set. The contrary would mean non-closed character of $\partial\ \Omega$. With the appearance of lacunas a part of diagonal Δ is cut out. The corresponding fixed points disappear. For everywhere dispersive Sinai billiard, lacuna absorbs all the diagonal and the mapping (1) will lack all fixed points. At a special configuration of such a border $\partial\ \Omega$ one can cut out cycles of higher order, $p \geq 2$.

In a topological way one can glue up the lacuna on the torus with a two-dimensional manifold. According to the rule of $\partial\ L$ bypass, it can be only a piece of a projective plane. This is directly connected with the projectivity of the billiard. On a projective plane, metric conceptions "inside" and "outside" of a closed region lose their sense. (For example, a closed curve and a right line that doesn't cross it on a plane may have common points after central projection onto the other plane.) That's why "forbidden" rays turn out to be involved into the general billiard flow. Such global motion takes place on non-oriented manifold.

On the projective plane the initial involution f also rules the motion of the rays. Almost every such ray (exceptional cases are of measure null) is continued to an ordinary billiard ray, further dynamics of which is known. As a result of further evolution, this ray after some time will return to the section of negative curvature $\partial\ \Omega$, corresponding to the lacuna under consideration. This is specified by the mentioned hyperbolicity of cycles that contain points on the concave border.

Being continued then to a classically inaccessible region (preserving the direction of motion), it would give a new position of the initial ray (phase point in the lacuna). A recurrent mapping appears. It is defines by one of the "compositions" f_k, included in the equation (3), the order k always depending on the coordinates of the initial ray (the initial point of the lacuna). The lacuna plays the role of a secant for the Poincaré section of the billiard flow. Similar evolution also takes place with other points of all lacunas. Only phase trajectories of ordinary billiard rays remain in this case "visible".

The condition of "connecting" for the inner rays, that are tangent to the concave region in the point $\vec{r}(z_3)$ and that cross $\partial\,\Omega$ in the points $\vec{r}(z_1)$, $\vec{r}(z_2)$ outside of it, defines the border of the corresponding lacuna

$$(\vec{r}(z_1) - \vec{r}(z_2),\ \vec{n}(z_3)) = 0\ ;\ (z_1, z_2) \in Z|\,L\ ;\ (z_1, z_3) \in \partial\,L\ ;\ (z_3, z_2) \in \partial\,L \quad ; \quad \hat{K}(z_3) < 0 \quad , \tag{8}$$

where $(.,.)$ is the scalar product of the vectors. Solving the equation (8) according to the theorem about an implicit function, we have $z_1 = \lambda_1(z_3);\ z_2 = \lambda_2(z_3)$. Excluding z_3, we come to the desired equation $z_1 = \lambda(z_2);\ \lambda = \lambda_1 \circ \lambda_2^{-1}$.

The discriminants D correspond to the zone of "stuck-together" trajectories or "non-continuable" trajectories that cross special (corner) points of $\partial\,\Omega$. That's why they appear in the billiards with straight regions of borders, $\hat{K} = 0$. Topologically the discriminants have a lacuna-like structure. Their border $\partial\,D$ is defined by

$$(\vec{r}(z_1) - \vec{r}(z_2),\ \vec{n}(z_2)) = 0\,;\ (z_1, z_2) \in Z|L \quad ; \quad \hat{K}(z_2) = 0 \quad . \tag{9}$$

It also can have explicit form $z_1 = \mu(z_2)$. The discriminants have regular shape – squares (Fig.) with a diagonal, which coincide with a part of $\Delta(z_1 = z_2)$ in the region corresponding to the straight-line component $\partial\,\Omega$.

Lacunas and discriminants make a principal property of a symmetric phase space. In fact, they are filled with the rays of the billiard that fell out of its ordinary dynamics. (At their passing it's easy to show that the billiard involution f breaks (on the first and the second arguments), the breaks being different from the factor of periodicity (mod 1) and are not removed when passing to a torus, $Z = I^2 \rightarrow Z = T^2$.) There are no such non-local elements in the phase space of Hamiltonian approach. At the same time, these formally hidden "topological" obstacles (Fig.) for the billiard flow to flow around, and the diagonal Δ, on which they arise, play an important role in the chaotic dynamics and must be included into the full description.

5. Billiard Kinetics: Invariant Distributions

The geometry of phase space structural elements depends on the form of the border $\partial\,\Omega$ and (or) involution f. Let us show that in the symmetrical approach not only dynamics but also kinetics of the billiard is connected with these characteristics. The kinetics becomes apparent in the case of chaotic billiard, whose deterministic trajectories (unique groups of successive points of reflection) have all the properties of random sequences in the asymptotic limit of infinitely large number of reflections. That requires statistic description of (two-dimensional) dynamic system in the manner of deterministic chaos conception [9].

In the mixed dynamics of a typical billiard both integrable and ergodic (as a rule, with intermixing) types of motion are present. In this case, the general characteristic of the trajectories distribution in the billiard is the invariant measure of the dynamic system. Absolutely continuous distributions are of the greatest physical interest. It's important for them to be individualized, i.e. depending on the geometry of a concrete billiard. That's why universal measures like conserved Liouville volume (Birkhov measure [1] for all billiards) are of little use here. A symmetric measure with density $\rho\,(z_1, z_2) = \rho\,(z_2, z_1)$ already possesses this individuality.

From the operator equation $B\rho = \rho$ for an invariant measure after transformations using piecewise monotony (5) we have

$$\rho(z_1, z_2) = \rho(z_2, z_1) = \rho(z_2, f(z_1, z_2)) \left(-\frac{\partial f(z_1, z_2)}{\partial z_1} \right) = \rho(z_2, f(z_1, z_2)) J(z_1, z_2) \quad . \tag{10}$$

In some cases (integrable billiards) one can find its explicit solution. Geometrically, ρ is a two-point density; it depends on the coordinates of two points on the border $\partial\,\Omega$. The topology of the direct product $Z \propto \partial\Omega \times \partial\Omega$ causes one to choose a special factorized solution, $\rho(z_1, z_2) = \omega(z_1)\,\omega(z_2)$. Instead of the expression (10) we get a functional equation for one-point plane $\omega(z)$:

$$\omega\,(z) = \omega\,(f)\,J(z,z');\;\; f = f(z,z') \quad \Leftrightarrow \quad \omega\,(z)\,dz = -\omega\,(f)\,df \quad , \tag{11}$$

written in total differentials. The factorization is coordinated with the symmetry of ρ and preserves its normalization $\|\omega\| = \int_0^1 \omega(z)\,dz = 1$.

The physical sense of $\omega(z)$ is an asymptotic plane of billiard flow reflection points (with coordinates $\vec{r}(z) \in \partial\Omega, \quad z \in I$). This is a truncate distribution in the sense that the dimension falls twice. It will be very useful in the description of physical characteristics in different billiard problems, for instance, the "probability" of ray escaping from a fixed place of resonator, wave-guide or detector. Besides, it is directly connected with the involution and geometry of the billiard. After integrating the differential relation (11) for ω we have

$$\left(\int\limits_{z_0}^{f} - \int\limits_{z_1}^{z_0}\right)\omega\,(z)\,dz = C(z_2)\,; \quad f = f(z_1, z_2) \quad , \tag{12}$$

where z_0 is an arbitrary initial point on $\partial\,\Omega$; $C(z)$ is the function to define. With different character of border $\partial\,\Omega C(z)$ has different forms. For everywhere convex billiard it's one can just use the diagonal condition $f(z,z) = z$, so $C(z) = 2\int_{z_0}^{z} \omega\,(z')\,dz'$. In the general case the border $\partial\,\Omega = \partial\Omega_+ \cup \partial\Omega_- \cup \partial\Omega_0$ contains regions of positive, $\partial\,\Omega_+$, negative, $\partial\,\Omega_-$, and zero, $\partial\,\Omega_0$, curvature. During the defining of $C(z)$ the solutions in symmetric "halves" of phase space over and under the diagonal Δ, that is, in the involutionally connected regions with coordinates (z_1, z_2) and $(f(z_1, z_2), z_2)$ are laced. In the presence of $\partial\,\Omega_-$ and $\partial\,\Omega_0$ components connecting takes place on the borders of corresponding lacunas and discriminants. Summing it up, let us set the border $\partial\,\Sigma$, that divides different symmetric components of ordinary cascade Σ (outside special zones)

$$(z_1, z_2) \in \partial\,\Sigma \quad \Leftrightarrow \quad z_1 = \Lambda(z_2) = \begin{cases} z_2\,, & (z_1, z_2) \in \Delta \\ \lambda(z_2), & (z_1, z_2) \in \partial\,L \\ \mu(z_2), & (z_1, z_2) \in \partial\,D \end{cases} \tag{13}$$

with known dependencies in the cases of lacunas and discriminants (see above). Let us note that in each half of the phase space $\Lambda(z)$ is a multi-valued function (the number of branches doesn't exceed the doubled number of $\partial\,\Omega_-$ and $\partial\,\Omega_0$ components, but self-intersections and multiple connection $\partial\,\Sigma$ are forbidden by the uniqueness of the flow. $\Lambda(z)$ is the functional of $\partial\,\Omega$form. Connecting on the border $\partial\,\Sigma$ gives us

$$\left(\int\limits_{z_0}^{f(\Lambda(z),\,z)} - \int\limits_{\Lambda(z)}^{z_0}\right)\omega\,(z')\,dz' = C(z) \quad \Rightarrow \quad \left(\int\limits_{\Lambda(z_2)}^{f(z_1,\,z_2)} - \int\limits_{z_1}^{f(\Lambda(z_2),\,z_2)}\right)\omega\,(z)\,dz = 0 \quad . \tag{14}$$

The dependence of the initial point z_0, as would be expected, falls out. The equation obtained lets one to restore billiard involution f on the one-point billiard distribution function $\omega$and vice versa. At the same time both functions are connected with the equation of border $\partial\,\Omega$ by the expressions (13) and (1). The billiard problem takes on a single meaning from dynamic, statistic and geometric points of view.

Direct solution for ω on f can be obtained by differentiation of Eq.(11)

$$\frac{d\ln\omega\,(f)}{df} = -\frac{\partial^2 f(z_1, z_2)/\partial z_1 \partial z_2}{\partial f(z_1, z_2)/\partial z_1 \; \partial f(z_1, z_2)/\partial z_2} \quad . \tag{15}$$

Having performed the change $z_1 \to f(z_1, z_2)$ and using the relation $f \circ f = id$ for calculating the derivatives, this equation for ω can be written as

$$[\ln \omega(z_1)]'_{z_1} = -\frac{f''_{z_1 z_2}}{{f'_{z_1}}^3 f'_{z_2}} + \frac{{f''}_{z_z^2}}{{f'_{z_1}}^4};\ f = f(z_1, z_2) \quad , \tag{16}$$

where strokes mark partial derivatives. The right part of the equation (16) shouldn't depend on z_2. This limits, on the one hand, the type of involution (not every involution can be billiard, that corresponds to the stressed role of projective transformations), and, on the other hand, the choice of phase space variables, allowing factorization of two-point measure to one-point ones. Covariant changes, leading to "factorizing" coordinates, correspond to the choice of a certain frame and way of border $\partial\,\Omega$ parameterization.

In the equations (10) and (16) the densities ρ and ω are uniquely defined by the involution of billiard f. The latter is uniquely defined by the border $\partial\,\Omega$equation, according to the representation (2). The invariant measures of the billiard become its individual characteristics. In a chaotic billiard they acquire the character of equilibrium statistic distributions.

So, on the whole billiard analysis in symmetrical coordinates shows that its main characteristics are uniquely connected with one another

$$\partial\,\Omega \to f(z_1, z_2) \to \rho(z_1, z_2) \leftrightarrow \omega\,(z)\ \to f(z_1, z_2) \to \partial\,\Omega \tag{17}$$

where arrows indicate passing from one object to another.

One of the most designing and old problems of statistical physics is finding out the transition from the reversibility of deterministic motion equations to irreversibility of statistic ones, see for ex. [10]. Generally accepted point of view is that irreversibility appears at roughening in the macroscopic description of the system on the kinetic stage of evolution and is connected with the fundamental principle of correlations unlinking. Here usually the problem of distribution functions calculation with given Hamiltonian (the equations of motion) is posed. In physical applications the inverse problem may also appear: to restore the dynamic law for a chaotic system (not necessarily of mechanic origin) with known statistic characteristics. It can be of special actuality for the system with a small number of freedom degrees.

Statistic irreversibility prevents the reverse of the "time arrow", but doesn't necessarily break the feedback of kinetic and dynamic. A remarkable peculiarity of the billiard is the possibility to solve direct as well as indirect problems. The form of the border $\partial\,\Omega$ defines involution, on which the invariant measure is calculated. And vice versa: the involution (that is, the dynamic of the billiard) is restored from the one-point distribution of reflections on the border. The border of the billiard can be restored by its involution [7]. By the way such closure is the consequence of geometric (projective) nature of the billiard.

6. Summary

In the conclusion let us note the characteristic properties of the symmetrical approach.

1. The symmetricity of the phase space. Equality of phase coordinates rights. Non-local character of the geometric elements involved into the dynamics.

2. Unification of billiards as reversible dynamic systems (mappings) with projective involution. Reversibility and projectivity of the billiard.

3. Geometric character of the phase space structure, taking into account billiard border properties (its diagonal, lacunas, discriminants). Universal character of symmetrical normal forms, describing the dynamics and local bifurcations in the neighbourhood of cycles and special zones of the billiard.

4. Individualization of invariant distributions defined by the form of the billiard. The reduction of the measure (one-point factorization) without loss of statistic description. The division of quick and slow variables is not necessary.

5. The unity of dynamics, kinetics and geometry of the billiard. The solution of direct and indirect problems on the restoration of involution, invariant measure and the form of the border.

Symmetric approach allows direct generalization on the multiply connected, multi-dimensional and other cases of different billiard border topology. The peculiarities named preserve here their key role.

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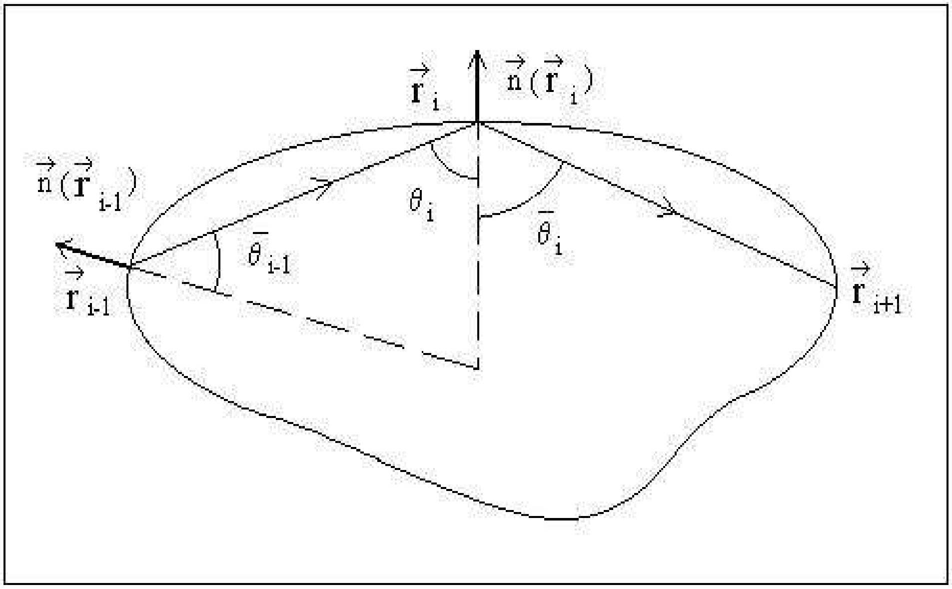
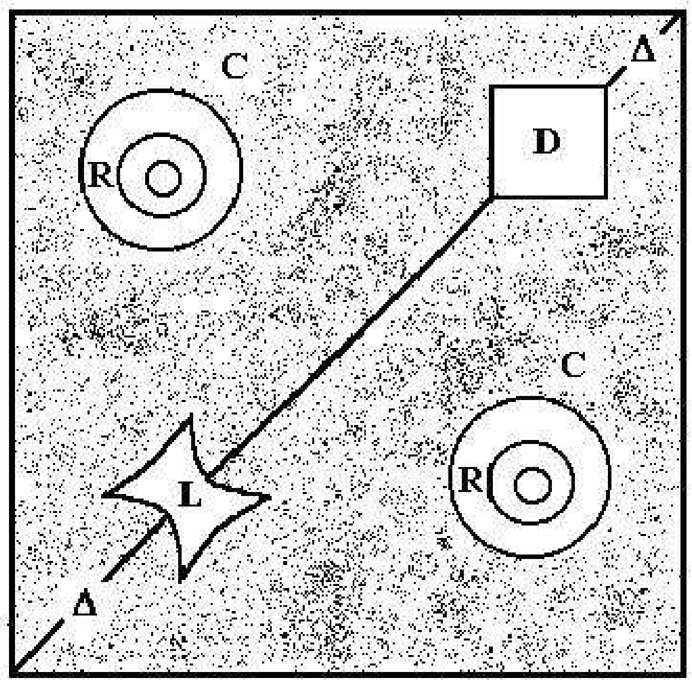
Figure

The geometry of a billiard and the schematic form of typical billiard symmetric phase space. Elliptic zones of regular motion R, a chaotic region C, the diagonal Δ, lacunas L and discriminants D.


$\vec{r}_i$
$\vec{n}(\vec{r}_i)$
$\vec{n}(\vec{r}_{i-1})$
θ_i
$\bar{\theta}_i$
$\bar{\theta}_{i-1}$
$\vec{r}_{i-1}$
$\vec{r}_{i+1}$


C
Δ
D
R
C
L
R
Δ